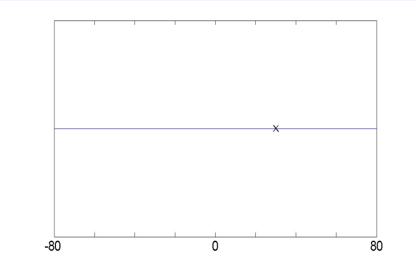
Information and uncertainty

Zdeněk Fabián
Ústav informatiky AVČR Praha

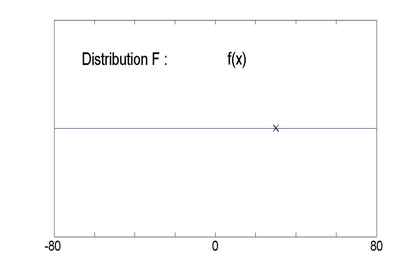
2018

Realization of continuous random variable



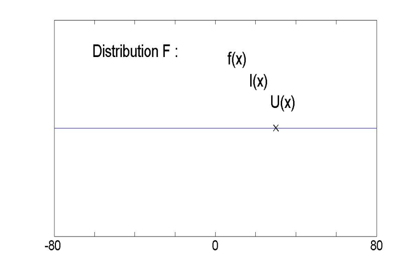
What amount of information/uncertainty it carries ?

Knowing model,



we ask :

Density of information $I(x)$?
Density of uncertainty $U(x)$?
Mean values I^* of $I(x)$ and U^* of $U(x)$?



$$I^* = \int_{\mathcal{X}} I(x) f(x)\, dx \qquad U^* = \int_{\mathcal{X}} U(x) f(x)\, dx$$

Response from information theory: $U^* = h_F$?

Mean uncertainty of distribution is differential entropy, the continuous equivalent of Shannon's entropy,

$$h_F = E(-\log f) = \int_{\mathcal{X}} \log \frac{1}{f(x)} f(x)\, dx$$

However, h_F could be negative: for normal distribution

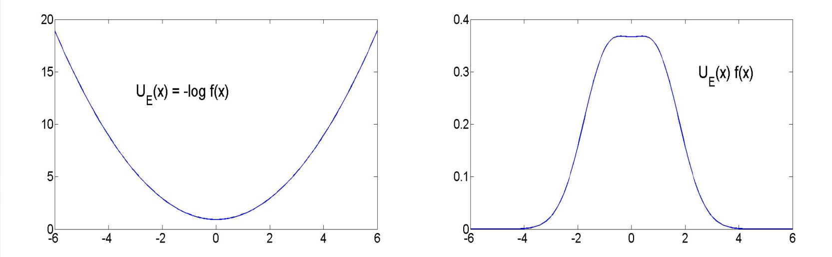
$$h_F = \log(\sqrt{2\pi e}\sigma)$$

Forming the differential entropy

standard normal :

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \qquad U_E(x) = -\log f(x) = \log \sqrt{2\pi} + \frac{1}{2}x^2$$

$$h_F = \int_{-\infty}^{\infty} U_E(x) f(x)\, dx$$



Response from statistics

Statistical information concept is Fisher information, defined with respect to parameters of parametric distributions

Let's have $f(x;\theta), \theta = (\theta_1, ..., \theta_m) \in \Theta \subseteq R^m$. Fisher (likelihood) scores are

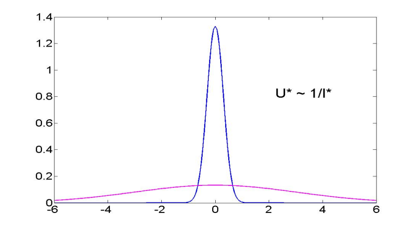
$$\psi_j(x) = \frac{\partial}{\partial\theta_j} \log f(x;\theta), \qquad j = 1, ..., m, \qquad \theta \in \Theta$$

Fisher information for θ_j is mean information

$$I(\theta_j) = E\psi_j^2$$

which carries x about θ_j

Expected relation between I^* and U^*



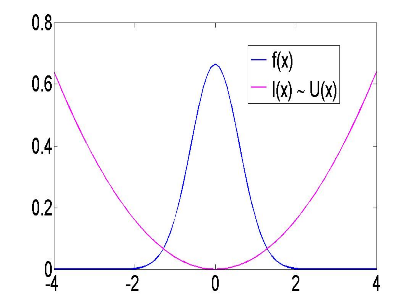
For normal distribution

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

Fisher information for μ $I^* = \frac{1}{\sigma^2}$ is FI for the mode

We suppose that $U^* = \frac{1}{I^*} = \sigma^2$

Expected relation between $I(x)$ and $U(x)$



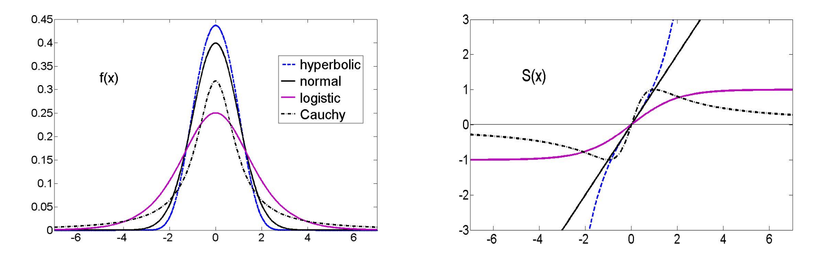
Score function of distributions with $\mathcal{X} = \mathcal{R}$

Score function

$$S(x) = -\frac{f'(x)}{f(x)}$$

For location model $F(x-\mu), \mu \in \mathcal{R}$ it holds that

$$\frac{\partial}{\partial \mu} \log f(x-\mu) = -\frac{1}{f(x-\mu)} \frac{d}{dx} f(x-\mu) = S(x-\mu)$$



Solution for normal distribution

- $f(x;\mu,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$

$$S(x)=\frac{x-\mu}{\sigma^2}$$

Solution for normal distribution

- $f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$

$$S(x) = \frac{x-\mu}{\sigma^2}$$

-

$$I(x) = S^2(x) = \frac{(x-\mu)^2}{\sigma^4} \qquad I^* = EI = \frac{1}{\sigma^2}$$

Solution for normal distribution

- $f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$

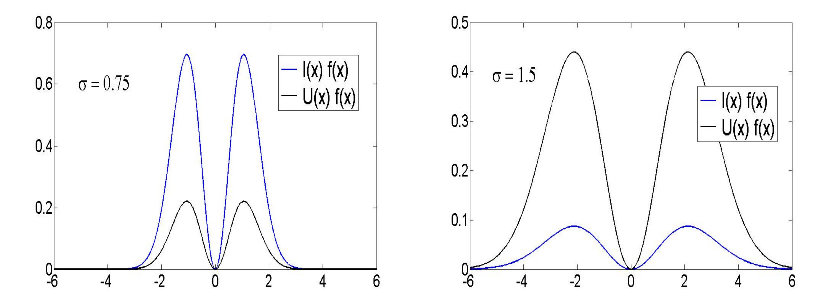
$$S(x) = \frac{x-\mu}{\sigma^2}$$

- $$I(x) = S^2(x) = \frac{(x-\mu)^2}{\sigma^4} \qquad I^* = EI = \frac{1}{\sigma^2}$$

$$U(x) = \frac{I(x)}{[EI]^2} = (x-\mu)^2 \qquad U^* = EU = \sigma^2$$

Forming of I^* and U^* for normal

$$I^* = \int_{-\infty}^{\infty} I(x)f(x)\, dx$$



Distributions with $U^* = I^* = 1$

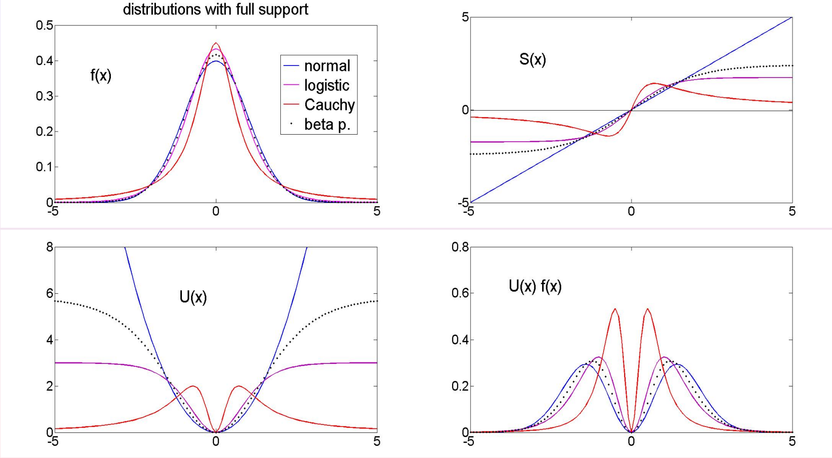


Table: Distributions on R

Set $u = \frac{x-\mu}{\sigma}$

name	$f(x)$	$S(x)$	U^*
Gumbel	$\frac{1}{\sigma} e^u e^{e^u}$	$\frac{1}{\sigma}(e^u - 1)$	σ^2
extreme val.	$\frac{1}{\sigma} e^{-u} e^{-e^{-u}}$	$\frac{1}{\sigma}(1 - e^{-u})$	σ^2
Cauchy	$\frac{1}{\sigma\pi} \frac{1}{1+u^2}$	$\frac{2u}{1+u^2}$	$2\sigma^2$
logistic	$\frac{1}{\sigma} \frac{u}{(u+1)^2}$	$\frac{1}{\sigma} \frac{u-1}{u+1}$	$3\sigma^2$
prot. gamma	$\frac{\gamma^\alpha}{\Gamma(\alpha)} e^{\alpha x} e^{-\gamma e^x}$	$\gamma e^x - \alpha$	$1/\alpha$
prot. beta	$\frac{1}{B(p,q)} \frac{e^{px}}{(e^x+1)^{p+q}}$	$\frac{pe^x - q}{e^x+1}$	$\frac{p+q+1}{pq}$

For $U^* = 1$, $I(x) = U(x)$. For $\sigma = const.$, normal is not a maximum uncertainty distribution

Distributions with partial support $\mathcal{X} \neq \mathcal{R}$

The presented approach is quite natural, but it is not used in statistics. The reason is, apparently, that the score function

$$S_F(x) = -\frac{f'(x)}{f(x)}$$

of distributions with partial support is out of use (cf. uniform distribution). We suggested the idea that one can consider distributions with partial support as transformed "prototypes" with full support, $F(x) = G(\eta(x))$.

Density of F is

$$f(x) = g(\eta(x))\eta'(x)$$

Define t-score of F as

$$T_F(x) = S_G(\eta(x))$$

Properties of t-scores

i/ t-score can be rewritten into

$$T_F(x) = -\frac{1}{f(x)} \frac{d}{dx} \left[\frac{1}{\eta'(x)} f(x) \right]$$

ii/ $\eta(x) = \log(x)$ if $\mathcal{X} = (0, \infty)$
$\eta(x) = \log \frac{x}{1-x}$ if $\mathcal{X} = (0, 1)$

iii/ $x^* : T_F(x) = 0$ is central value: it is the "image" of the mode of the "prototype" G

iv/ Let $F : F(x; \theta)$ and $\theta = (\tau, \theta_2, ...)$ Then Fisher score for $\tau \sim T_F(x; \tau)$

$$S(x) = const * T_F(x)$$

information density

$$I(x) = S^2(x)$$

uncertainty density

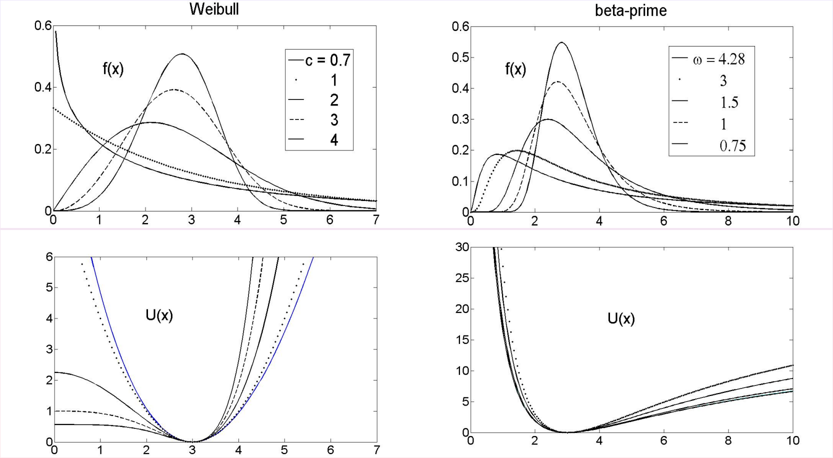
$$U(x) = \frac{I(x)}{[EI]^2}$$

$$U^* = \frac{1}{I^*}$$

Examples from $\mathcal{X} = (0, \infty)$

F	$f(x)$	$S_F(x)$	x^*	U^*
lognormal	$\frac{c}{\sqrt{2\pi}x} e^{-\frac{1}{2}\log^2(\frac{x}{\tau})^c}$	$\frac{c}{\tau}\log(\frac{x}{\tau})^c$	τ	$\frac{\tau^2}{c^2}$
exponential	$\frac{1}{\tau}e^{-x/\tau}$	$\frac{1}{\tau}(\frac{x}{\tau}-1)$	τ	τ^2
Weibull	$\frac{c}{x}(\frac{x}{\tau})^c e^{-(\frac{x}{\tau})^c}$	$\frac{c}{\tau}[(\frac{x}{\tau})^c-1]$	τ	$\frac{\tau^2}{c^2}$
loglogistic	$\frac{c}{\tau}\frac{(x/\tau)^{c-1}}{[(x/\tau)^c+1]^2}$	$\frac{c}{\tau}\frac{(x/\tau)^c-1}{(x/\tau)^c+1}$	τ	$\frac{3\tau^2}{c^2}$
gamma	$\frac{\gamma^\alpha}{x\Gamma(\alpha)}x^\alpha e^{-\gamma x}$	$\frac{\gamma^2}{\alpha}(x-x^*)$	$\frac{\alpha}{\gamma}$	$\frac{\alpha}{\gamma^2}$
beta-prime	$\frac{1}{B(p,q)}\frac{x^{p-1}}{(x+1)^{p+q}}$	$\frac{q^2}{p}\frac{x-x^*}{x+1}$	$\frac{p}{q}$	$\frac{p(p+q+1)}{q^3}$

Weibull and beta-prime $U_W^* = 9/c^2 \quad U_{BP}^* = \omega^2$



ω and differential entropy

F	e^{h_F}	$\omega = \sqrt{U^*}$
normal	$\sqrt{2\pi e}\sigma$	σ
Cauchy	$4\pi\sigma$	$\sqrt{2}\sigma$
exponential	$e\tau$	τ
lognormal	$\sqrt{2\pi e}\,\tau/c$	τ/c
Weibull	$e^{(1+\epsilon(1-1/c))}\,\tau/c$	τ/c
uniform$_{(a,b)}$	$b-a$	$\frac{\sqrt{3}}{4}(b-a)$

Reference

Fabián Z. (2016): Score function of distribution and revival of the moment method. Communication in Statistics, Theory-Methods 45: 1118-1136.

Thank you for your attention